

# TIGHT RELAXATIONS FOR POLYNOMIAL OPTIMIZATION AND LAGRANGE MULTIPLIER EXPRESSIONS

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**ABSTRACT.** This paper proposes tight semidefinite relaxations for polynomial optimization. The optimality conditions are investigated. We show that generally Lagrange multipliers can be expressed as polynomial functions in decision variables over the set of critical points. The polynomial expressions can be determined by linear equations. Based on these expressions, new Lasserre type semidefinite relaxations are constructed for solving polynomial optimization. We show that the hierarchy of new relaxations has finite convergence, or equivalently, the new relaxations are tight for a finite relaxation order.

## 1. INTRODUCTION

A general class of optimization problems is

$$(1.1) \quad \begin{cases} f_{\min} := \min & f(x) \\ s.t. & c_i(x) = 0 \ (i \in \mathcal{E}), \\ & c_j(x) \geq 0 \ (j \in \mathcal{I}), \end{cases}$$

where  $f$  and all  $c_i, c_j$  are polynomials in  $x := (x_1, \dots, x_n)$ , the real decision variable. The  $\mathcal{E}$  and  $\mathcal{I}$  are two disjoint finite index sets of constraining polynomials. Lasserre's relaxations [16] are generally used for solving (1.1) globally, i.e., to find the global minimum value  $f_{\min}$  and minimizer(s) if any. The convergence of Lasserre's relaxations is related to optimality conditions.

**1.1. Optimality conditions.** A general introduction of optimality conditions in nonlinear programming can be found in [1, Section 3.3]. Let  $u$  be a local minimizer of (1.1). Denote the index set of active constraints

$$(1.2) \quad J(u) := \{i \in \mathcal{E} \cup \mathcal{I} \mid c_i(u) = 0\}.$$

If the *constraint qualification condition (CQC)* holds at  $u$ , i.e., the gradients  $\nabla c_i(u)$  ( $i \in J(u)$ ) are linearly independent ( $\nabla$  denotes the gradient), then there exist Lagrange multipliers  $\lambda_i$  ( $i \in \mathcal{E} \cup \mathcal{I}$ ) satisfying

$$(1.3) \quad \nabla f(u) = \sum_{i \in \mathcal{E} \cup \mathcal{I}} \lambda_i \nabla c_i(u),$$

$$(1.4) \quad c_i(u) = 0 \ (i \in \mathcal{E}), \quad \lambda_j c_j(u) = 0 \ (j \in \mathcal{I}),$$

$$(1.5) \quad c_j(u) \geq 0 \ (j \in \mathcal{I}), \quad \lambda_j \geq 0 \ (j \in \mathcal{I}).$$

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The second equation in (1.4) is called the *complementarity condition*. If  $\lambda_j + c_j(u) > 0$  for all  $j \in \mathcal{I}$ , the *strict complementarity condition* (SCC) is said to hold. For the  $\lambda_i$ 's satisfying (1.3)-(1.5), the associated Lagrange function is

$$\mathcal{L}(x) := f(x) - \sum_{i \in \mathcal{E} \cup \mathcal{I}} \lambda_i c_i(x).$$

Under the constraint qualification condition, the *second order necessary condition* (SONC) holds at  $u$ , i.e., ( $\nabla^2$  denotes the Hessian)

$$(1.6) \quad v^T \left( \nabla^2 \mathcal{L}(u) \right) v \geq 0 \quad \text{for all } v \in \bigcap_{i \in J(u)} \nabla c_i(u)^\perp.$$

Here,  $\nabla c_i(u)^\perp$  is the orthogonal complement of  $\nabla c_i(u)$ . If it further holds that

$$(1.7) \quad v^T \left( \nabla^2 \mathcal{L}(u) \right) v > 0 \quad \text{for all } 0 \neq v \in \bigcap_{i \in J(u)} \nabla c_i(u)^\perp,$$

then the *second order sufficient condition* (SOSC) is said to hold. If the constraint qualification condition holds at  $u$ , then (1.3), (1.4) and (1.6) are necessary conditions for  $u$  to be a local minimizer. If (1.3), (1.4), (1.7) and the strict complementarity condition hold, then  $u$  is a strict local minimizer.

**1.2. Some existing work.** Under the archimedean condition (see §2), the hierarchy of Lasserre's relaxations converges asymptotically [16]. Moreover, in addition to the archimedeaness, if the constraint qualification, strict complementarity, and second order sufficient conditions hold at every global minimizer, then the Lasserre's hierarchy converges in finitely many steps [31]. For convex polynomial optimization, the Lasserre's hierarchy has finite convergence under the strict convexity or sos-convexity condition [7, 19]. For unconstrained polynomial optimization, the standard sum of squares relaxation was proposed in [33]. When the equality constraints define a finite set, the Lasserre's hierarchy also has finite convergence, as shown in [17, 23, 29]. Recently, a bounded degree hierarchy of relaxations was proposed for polynomial optimization [22]. General introductions to polynomial optimization and moment problems can be found in the books and surveys [20, 21, 24, 25, 37]. Lasserre's relaxations provide lower bounds for the minimum value. There also exist methods that compute upper bounds [8, 18]. A convergence rate analysis for such upper bounds is given in [9, 10]. When a polynomial optimization problem does not have minimizers (i.e., the infimum is not achievable), there are relaxation methods for computing the infimum [36, 40].

A new type of Lasserre relaxations, based on Jacobian representations, were recently proposed in [28]. The hierarchy of such relaxations always has finite convergence, when the feasible sets are nonsingular. When there are only equality constraints  $c_1(x) = \dots = c_m(x) = 0$ , the method needs the maximal minors of the matrix

$$\begin{bmatrix} \nabla f(x) & \nabla c_1(x) & \dots & \nabla c_m(x) \end{bmatrix}.$$

When there are inequality constraints, it requires to enumerate all possibilities of active constraints. The method in [28] is expensive when there are a lot of constraints. For unconstrained optimization, it is reduced to the gradient sum of squares relaxations in [26].

**1.3. New contributions.** When Lasserre's relaxations are used to solve polynomial optimization, the following issues are of concerns:

- The convergence depends on the archimedean condition (see §2), which is satisfied only if the feasible set is compact. If the set is noncompact, how can we get convergent relaxations?
- The cost of Lasserre's relaxations depends significantly on the relaxation order. For a fixed order, can we construct tighter relaxations than the standard ones?
- When the convergence of Lasserre's relaxations is slow, can we construct new relaxations whose convergence is faster?
- When the optimality conditions fail to hold, the Lasserre's hierarchy might not have finite convergence. Can we construct a new hierarchy of stronger relaxations that also has finite convergence for such cases?

This paper discusses the above issues. We construct tighter relaxations by using optimality conditions. In (1.3)-(1.4), under the constraint qualification condition, the Lagrange multipliers  $\lambda_i$  are uniquely determined by  $u$ . Note that  $(u, \lambda)$  is a solution to the polynomial system in  $(x, \lambda)$ :

$$(1.8) \quad \sum_{i \in \mathcal{E} \cup \mathcal{I}} \lambda_i \nabla c_i(x) = \nabla f(x), \quad c_i(x) = \lambda_j c_j(x) = 0 \ (i \in \mathcal{E}, j \in \mathcal{I}).$$

A point  $x$  satisfying (1.8) is called a *critical point*, and such  $(x, \lambda)$  is called a critical pair. In (1.8), once  $x$  is known,  $\lambda$  can be determined by linear equations. Generally, the value of  $x$  is not known. One can try to express  $\lambda$  as a rational function in  $x$ . Suppose  $\mathcal{E} \cup \mathcal{I} = \{1, \dots, m\}$  and denote

$$G(x) := [\nabla c_1(x) \ \cdots \ \nabla c_m(x)].$$

When  $m \leq n$  and  $\text{rank } G(x) = m$ , we can get the rational expression

$$(1.9) \quad \lambda = (G(x)^T G(x))^{-1} G(x)^T \nabla f(x).$$

Typically, the matrix inverse  $(G(x)^T G(x))^{-1}$  is expensive for usage. The denominator  $\det(G(x)^T G(x))$  is typically a high degree polynomial. When  $m > n$ ,  $G(x)^T G(x)$  is always singular and we cannot express  $\lambda$  as in (1.9).

Do there exist polynomials  $p_i$  ( $i \in \mathcal{E} \cup \mathcal{I}$ ) such that each

$$(1.10) \quad \lambda_i = p_i(x)$$

for all  $(x, \lambda)$  satisfying (1.8)? If they exist, then we can do:

- The polynomial system (1.8) can be simplified to

$$(1.11) \quad \sum_{i \in \mathcal{E} \cup \mathcal{I}} p_i(x) \nabla c_i(x) = \nabla f(x), \quad c_i(x) = p_j(x) c_j(x) = 0 \ (i \in \mathcal{E}, j \in \mathcal{I}).$$

- For each  $j \in \mathcal{I}$ , the sign condition  $\lambda_j \geq 0$  is equivalent to

$$(1.12) \quad p_j(x) \geq 0.$$

The new conditions (1.11) and (1.12) only use the decision variable  $x$ , but not  $\lambda$ . They can be used to construct tighter relaxations for solving (1.1).

When do there exist polynomials  $p_i$  satisfying (1.10)? If they exist, how can we compute them? How can we use them to construct tighter relaxations? Do the new relaxations have advantages over the old ones? These questions are the main topics of this paper. Our major results are:

- We show that the polynomials  $p_i$  satisfying (1.10) exist for nonsingular constraints. Moreover, they can be determined by linear equations.
- Using the new conditions (1.11)-(1.12), we can construct tight relaxations for solving (1.1). To be more precise, we construct a hierarchy of new relaxations, which has finite convergence. This is true even if the feasible set is noncompact and/or the optimality conditions fail to hold.
- For every relaxation order, the new relaxations are tighter than the standard ones in the prior work.

The paper is organized as follows. Section 2 reviews some basics in polynomial optimization. Section 3 constructs new relaxations and proves their tightness. Section 4 characterizes when the polynomials  $p_i$ 's satisfying (1.10) exist and shows how to determine them, for polyhedral constraints. Section 5 discusses the case of general nonlinear constraints. Section 6 gives examples of using the new relaxations. Section 7 discusses some related issues.

## 2. PRELIMINARIES

**Notation** The symbol  $\mathbb{N}$  (resp.,  $\mathbb{R}$ ,  $\mathbb{C}$ ) denotes the set of nonnegative integral (resp., real, complex) numbers. The symbol  $\mathbb{R}[x] := \mathbb{R}[x_1, \dots, x_n]$  denotes the ring of polynomials in  $x := (x_1, \dots, x_n)$  with real coefficients. The  $\mathbb{R}[x]_d$  stands for the set of real polynomials with degrees  $\leq d$ . Denote

$$\mathbb{N}_d^n := \{\alpha := (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n \mid |\alpha| := \alpha_1 + \dots + \alpha_n \leq d\}.$$

For a polynomial  $p$ ,  $\deg(p)$  denotes its total degree. For  $t \in \mathbb{R}$ ,  $\lceil t \rceil$  denotes the smallest integer  $\geq t$ . For an integer  $k > 0$ , denote  $[k] := \{1, 2, \dots, k\}$ . For  $x = (x_1, \dots, x_n)$  and  $\alpha = (\alpha_1, \dots, \alpha_n)$ , denote

$$x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}, \quad [x]_d := \begin{bmatrix} 1 & x_1 & \cdots & x_n & x_1^2 & x_1 x_2 & \cdots & x_n^d \end{bmatrix}^T.$$

The superscript  $^T$  denotes the transpose of a matrix/vector. The  $e_i$  denotes the  $i$ th standard unit vector, while  $e$  denotes the vector of all ones. The  $I_m$  denotes the  $m$ -by- $m$  identity matrix. By writing  $X \succeq 0$  (resp.,  $X \succ 0$ ), we mean that  $X$  is a symmetric positive semidefinite (resp., positive definite) matrix. For matrices  $X_1, \dots, X_r$ ,  $\text{diag}(X_1, \dots, X_r)$  denotes the block diagonal matrix whose diagonal blocks are  $X_1, \dots, X_r$ . In particular, for a vector  $a$ ,  $\text{diag}(a)$  denotes the diagonal matrix whose diagonal entry vector is  $a$ . For a function  $f$  in  $x$ ,  $f_{x_i}$  denotes the partial derivative of  $f$  with respect to  $x_i$ .

We review some basics in computational algebra and polynomial optimization. They could be found in [4, 20, 21, 24, 25]. An ideal  $I$  of  $\mathbb{R}[x]$  is a subset such that  $I \cdot \mathbb{R}[x] \subseteq I$  and  $I + I \subseteq I$ . For a tuple  $h = (h_1, \dots, h_m)$  of polynomials,  $\text{Ideal}(h)$  denotes the smallest ideal containing all  $h_i$ , which is the set

$$h_1 \cdot \mathbb{R}[x] + \cdots + h_m \cdot \mathbb{R}[x].$$

The  $2k$ th *truncation* of  $\text{Ideal}(h)$  is the set

$$\text{Ideal}(h)_{2k} := h_1 \cdot \mathbb{R}[x]_{2k - \deg(h_1)} + \cdots + h_m \cdot \mathbb{R}[x]_{2k - \deg(h_m)}.$$

For an ideal  $I$ , its complex and real varieties are respectively defined as

$$\mathcal{V}_{\mathbb{C}}(I) := \{v \in \mathbb{C}^n \mid p(v) = 0 \forall p \in I\}, \quad \mathcal{V}_{\mathbb{R}}(I) := \mathcal{V}_{\mathbb{C}}(I) \cap \mathbb{R}^n.$$

A polynomial  $\sigma$  is said to be a sum of squares (SOS) if  $\sigma = s_1^2 + \dots + s_k^2$  for some polynomials  $s_1, \dots, s_k \in \mathbb{R}[x]$ . The set of all SOS polynomials in  $x$  is denoted as  $\Sigma[x]$ . For a degree  $d$ , denote the truncation

$$\Sigma[x]_d := \Sigma[x] \cap \mathbb{R}[x]_d.$$

For a tuple  $g = (g_1, \dots, g_t)$ , its *quadratic module* is the set

$$\text{Qmod}(g) := \Sigma[x] + g_1 \cdot \Sigma[x] + \dots + g_t \cdot \Sigma[x].$$

The  $2k$ th truncation of  $\text{Qmod}(g)$  is the set

$$\text{Qmod}(g)_{2k} := \Sigma[x]_{2k} + g_1 \cdot \Sigma[x]_{2k - \deg(g_1)} + \dots + g_t \cdot \Sigma[x]_{2k - \deg(g_t)}.$$

For convenience of notation, we denote

$$(2.1) \quad \begin{cases} \text{IQ}(h, g) &:= \text{Idea}(h) + \text{Qmod}(g), \\ \text{IQ}(h, g)_{2k} &:= \text{Idea}(h)_{2k} + \text{Qmod}(g)_{2k}. \end{cases}$$

The set  $\text{IQ}(h, g)$  is said to be *archimedean* if there exists  $p \in \text{IQ}(h, g)$  such that  $p(x) \geq 0$  defines a compact set. If  $\text{IQ}(h, g)$  is archimedean, then the set

$$K := \{x \in \mathbb{R}^n \mid h(x) = 0, g(x) \geq 0\}$$

must be compact. Conversely, if  $K$  is compact, say,  $K \subseteq B(0, R)$  (the ball centered at 0 with radius  $R$ ), then  $\text{IQ}(h, (g, R^2 - x^T x))$  is always archimedean and  $h = 0$ ,  $(g, R^2 - x^T x) \geq 0$  give the same set  $K$ .

**Theorem 2.1** (Putinar, [34]). *Let  $h, g$  be tuples of polynomials in  $\mathbb{R}[x]$ . Let  $K$  be as above. Assume  $\text{IQ}(h, g)$  is archimedean. If a polynomial  $f \in \mathbb{R}[x]$  is positive on  $K$ , then  $f \in \text{IQ}(h, g)$ .*

Interestingly, if  $f$  is only nonnegative on  $K$  but some standard optimality conditions hold, then we still have  $f \in \text{IQ}(h, g)$  [31].

Let  $\mathbb{R}_d^{\mathbb{N}}_d$  be the space of real sequences indexed by  $\alpha \in \mathbb{N}_d^n$ . A vector in  $\mathbb{R}_d^{\mathbb{N}}_d$  is called a *truncated multi-sequence* (tms) of degree  $d$ . A tms  $y := (y_\alpha)_{\alpha \in \mathbb{N}_d^n}$  defines the Riesz functional  $\mathcal{R}_y$  acting on  $\mathbb{R}[x]_d$  as

$$(2.2) \quad \mathcal{R}_y \left( \sum_{\alpha \in \mathbb{N}_d^n} f_\alpha x^\alpha \right) := \sum_{\alpha \in \mathbb{N}_d^n} f_\alpha y_\alpha.$$

For  $f \in \mathbb{R}[x]_d$  and  $y \in \mathbb{R}_d^{\mathbb{N}}_d$ , we denote

$$(2.3) \quad \langle f, y \rangle := \mathcal{R}_y(f).$$

Let  $q \in \mathbb{R}[x]_{2k}$ . The  $k$ th *localizing matrix* of  $q$ , generated by a tms  $y \in \mathbb{R}_{2k}^{\mathbb{N}}_{2k}$ , is the symmetric matrix  $L_q^{(k)}(y)$  such that

$$(2.4) \quad \text{vec}(a_1)^T \left( L_q^{(k)}(y) \right) \text{vec}(a_2) = \mathcal{R}_y(q a_1 a_2)$$

for all  $a_1, a_2 \in \mathbb{R}[x]_{k - \lceil \deg(q)/2 \rceil}$ . (The  $\text{vec}(a_i)$  denotes the coefficient vector of  $a_i$ .) When  $q = 1$  (the constant one polynomial),  $L_q^{(k)}(y)$  is called a *moment matrix* and we denote

$$(2.5) \quad M_k(y) := L_1^{(k)}(y).$$

The columns and rows of  $L_q^{(k)}(y)$ , as well as  $M_k(y)$ , are indexed by  $\alpha \in \mathbb{N}^n$  with  $2|\alpha| + \deg(q) \leq 2k$ . When  $q = (q_1, \dots, q_r)$  is a tuple of polynomials, we define

$$(2.6) \quad L_q^{(k)}(y) := \text{diag} \left( L_{q_1}^{(k)}(y), \dots, L_{q_r}^{(k)}(y) \right),$$

a block diagonal matrix. For the polynomial tuples  $h, g$  as above, the set

$$(2.7) \quad \mathcal{S}(h, g)_{2k} := \left\{ y \in \mathbb{R}^{\mathbb{N}_n^{2k}} \mid L_h^{(k)}(y) = 0, L_g^{(k)}(y) \succeq 0 \right\}$$

is a spectrahedral cone in  $\mathbb{R}^{\mathbb{N}_n^{2k}}$ . The set  $\text{IQ}(h, g)_{2k}$  is also a convex cone in  $\mathbb{R}[x]_{2k}$ . The dual cone of  $\text{IQ}(h, g)_{2k}$  is precisely  $\mathcal{S}(h, g)_{2k}$  [21, 24, 32]. This is because for all  $p \in \text{IQ}(h, g)_{2k}$  and for all  $y \in \mathcal{S}(h, g)_{2k}$ , we have  $\langle p, y \rangle \geq 0$ .

### 3. THE CONSTRUCTION OF TIGHT RELAXATIONS

Consider the polynomial optimization problem (1.1). Let

$$\lambda := (\lambda_i)_{i \in \mathcal{E} \cup \mathcal{I}}$$

be the vector of Lagrange multipliers. Denote the set

$$(3.1) \quad \mathcal{K} := \left\{ (x, \lambda) \in \mathbb{R}^n \times \mathbb{R}^{\mathcal{E} \cup \mathcal{I}} \mid \begin{array}{l} c_i(x) = \lambda_j c_j(x) = 0 \ (i \in \mathcal{E}, j \in \mathcal{I}) \\ \nabla f(x) = \sum_{i \in \mathcal{E} \cup \mathcal{I}} \lambda_i \nabla c_i(x) \end{array} \right\}.$$

Each point in  $\mathcal{K}$  is called a critical pair. The projection

$$(3.2) \quad \mathcal{K}_c := \{u \mid (u, \lambda) \in \mathcal{K}\}$$

is the set of real critical points for (1.1). To construct tight relaxations for solving (1.1), we need the following assumption for Lagrange multipliers.

**Assumption 3.1.** *For each  $i \in \mathcal{E} \cup \mathcal{I}$ , there exists a polynomial  $p_i \in \mathbb{R}[x]$  such that for all  $(x, \lambda) \in \mathcal{K}$  it holds that*

$$\lambda_i = p_i(x).$$

Assumption 3.1 is generally satisfied. For the following cases, we can express  $\lambda$  as a polynomial function in  $x$  explicitly.

- (Simplex) For the simplex  $\{e^T x - 1 = 0, x_1 \geq 0, \dots, x_n \geq 0\}$ , it corresponds to that  $\mathcal{E} = \{0\}$ ,  $\mathcal{I} = [n]$ ,  $c_0(x) = e^T x - 1$ ,  $c_j(x) = x_j$  ( $j \in [n]$ ). The Lagrange multipliers can be expressed as

$$(3.3) \quad \lambda_0 = x^T \nabla f(x), \quad \lambda_j = f_{x_j} - x^T \nabla f(x) \quad (j \in [n]).$$

- (Hypercube) For the hypercube  $[-1, 1]^n$ , it corresponds to that  $\mathcal{E} = \emptyset$ ,  $\mathcal{I} = [n]$  and each  $c_j(x) = 1 - x_j^2$ . We can show that

$$(3.4) \quad \lambda_j = -\frac{1}{2} x_j f_{x_j} \quad (j \in [n]).$$

- (Ball or sphere) The constraint is  $1 - x^T x = 0$  or  $1 - x^T x \geq 0$ . It corresponds to that  $\mathcal{E} \cup \mathcal{I} = \{1\}$  and  $c_1 = 1 - x^T x$ . We have

$$(3.5) \quad \lambda_1 = -\frac{1}{2} x^T \nabla f(x).$$

- (Triangular constraints) Suppose  $\mathcal{E} \cup \mathcal{I} = \{1, \dots, m\}$  and each

$$c_i(x) = \tau_i x_i + q_i(x_{i+1}, \dots, x_n)$$

for some polynomials  $q_i \in \mathbb{R}[x_{i+1}, \dots, x_n]$  and scalars  $\tau_i \neq 0$ . The matrix  $T(x)$ , consisting of the first  $m$  rows of  $[\nabla c_1(x), \dots, \nabla c_m(x)]$ , is an invertible lower triangular matrix with constant diagonal entries. Then,

$$\lambda = T(x)^{-1} \cdot [f_{x_1} \quad \dots \quad f_{x_m}]^T.$$

Note that the inverse  $T(x)^{-1}$  is also a matrix polynomial.

For more general constraints, we can also express  $\lambda$  as a polynomial function in  $x$  on the set  $\mathcal{K}_c$ . This will be discussed in §4 and §5.

For the polynomials  $p_i$  as in Assumption 3.1, denote

$$(3.6) \quad \phi := \left( \nabla f - \sum_{i \in \mathcal{E} \cup \mathcal{I}} p_i \nabla c_i, (p_j c_j)_{j \in \mathcal{I}} \right), \quad \psi := (p_j)_{j \in \mathcal{I}}.$$

When the minimum value  $f_{\min}$  of (1.1) is achieved at a critical point, (1.1) is equivalent to the problem

$$(3.7) \quad \begin{cases} f_c := \min & f(x) \\ \text{s.t.} & c_{eq}(x) = 0, c_{in}(x) \geq 0, \\ & \phi(x) = 0, \psi(x) \geq 0. \end{cases}$$

We can apply Lasserre relaxations to solve it. For an integer  $k > 0$  (called the *relaxation order*), the  $k$ th order Lasserre's relaxation for (3.7) is

$$(3.8) \quad \begin{cases} f'_k := \min & \langle f, y \rangle \\ \text{s.t.} & \langle 1, y \rangle = 1, M_k(y) \succeq 0 \\ & L_{c_{eq}}^{(k)}(y) = 0, L_{c_{in}}^{(k)}(y) \succeq 0, \\ & L_{\phi}^{(k)}(y) = 0, L_{\psi}^{(k)}(y) \succeq 0, y \in \mathbb{R}^{\mathbb{N}_{2k}^n}. \end{cases}$$

Since  $x^0 = 1$  (the constant one polynomial), the condition  $\langle 1, y \rangle = 1$  means that  $(y)_0 = 1$ . Its dual optimization problem is

$$(3.9) \quad \begin{cases} f_k := \max & \gamma \\ \text{s.t.} & f - \gamma \in \text{IQ}(c_{eq}, c_{in})_{2k} + \text{IQ}(\phi, \psi)_{2k}. \end{cases}$$

We refer to §2 for the notation used in (3.8)-(3.9). They are equivalent to some semidefinite programs (SDPs), so they can be solved by SDP solvers (e.g., **SeDuMi** [38]). For  $k = 1, 2, \dots$ , we get a hierarchy of Lasserre relaxations. In (3.8)-(3.9), if we remove the usage of  $\phi$  and  $\psi$ , they are reduced to standard Lasserre relaxations in [16]. So, (3.8)-(3.9) are stronger relaxations.

By the construction of  $\phi$  as in (3.6), Assumption 3.1 implies that

$$\mathcal{K}_c = \{u \in \mathbb{R}^n : c_{eq}(u) = 0, \phi(u) = 0\}.$$

By Lemma 3.3 of [6],  $f$  achieves only finitely many values on  $\mathcal{K}_c$ , say,

$$(3.10) \quad v_1 < \dots < v_N.$$

A point  $u \in \mathcal{K}_c$  may not be feasible for (3.7), i.e.,  $c_{in}(u) \not\geq 0$  or  $\psi(u) \not\geq 0$ . In applications, we are often interested in the optimal value  $f_c$  of (3.7). When (3.7) is infeasible, by convention, we set

$$f_c = +\infty.$$

When the optimal value  $f_{\min}$  of (1.1) is achieved at a critical point,  $f_c = f_{\min}$ . This is the case if the feasible set is compact (or  $f$  is coercive) and the constraint qualification condition holds. As in [16], one can show that

$$(3.11) \quad f_k \leq f'_k \leq f_c$$

for all  $k$ . Moreover,  $\{f_k\}$  and  $\{f'_k\}$  are both monotonically increasing. If for some order  $k$  it occurs that

$$f_k = f'_k = f_c,$$

then the  $k$ th order Lasserre's relaxation is said to be *tight* (or *exact*).

**3.1. Tightness of the relaxations.** Let  $c_{in}, \psi, \mathcal{K}_c, f_c$  be as above. We refer to §2 for the notation  $\text{Qmod}$ . We begin with a general assumption.

**Assumption 3.2.** *There exists  $\rho \in \text{Qmod}(c_{in}, \psi)$  such that if  $u \in \mathcal{K}_c$  and  $f(u) < f_c$ , then  $\rho(u) < 0$ .*

In Assumption 3.2, the hypersurface  $\rho(x) = 0$  separates feasible and nonfeasible critical points. Clearly, if  $u \in \mathcal{K}_c$  is a feasible point for (3.7), then  $c_{in}(u) \geq 0$  and  $\psi(u) \geq 0$ , and hence  $\rho(u) \geq 0$ . Assumption 3.2 generally holds. For instance, it is satisfied for the following general cases.

- a) When there are no inequality constraints,  $c_{in}$  and  $\psi$  are empty tuples. Then,  $\text{Qmod}(c_{in}, \psi) = \Sigma[x]$  and Assumption 3.2 is satisfied for  $\rho = 0$ .
- b) Suppose the set  $\mathcal{K}_c$  is finite, say,  $\mathcal{K}_c = \{u_1, \dots, u_D\}$ , and

$$f(u_1), \dots, f(u_{t-1}) < f_c \leq f(u_t), \dots, f(u_D).$$

Let  $\ell_1, \dots, \ell_D$  be real interpolating polynomials such that  $\ell_i(u_j) = 1$  for  $i = j$  and  $\ell_i(u_j) = 0$  for  $i \neq j$ . For each  $i = 1, \dots, t$ , there must exist  $j_i \in \mathcal{I}$  such that  $c_{j_i}(u_i) < 0$ . Then, the polynomial

$$(3.12) \quad \rho := \sum_{i < t} \frac{-1}{c_{j_i}(u_i)} c_{j_i}(x) \ell_i(x)^2 + \sum_{i \geq t} \ell_i(x)^2$$

satisfies Assumption 3.2.

- c) For each  $x$  with  $f(x) = v_i < f_c$ , at least one of the constraints  $c_j(x) \geq 0, p_j(x) \geq 0 (j \in \mathcal{I})$  is violated. Suppose for each critical value  $v_i < f_c$ , there exists  $g_i \in \{c_j, p_j\}_{j \in \mathcal{I}}$  such that

$$g_i < 0 \quad \text{on} \quad \mathcal{K}_c \cap \{f(x) = v_i\}.$$

Let  $\varphi_1, \dots, \varphi_N$  be real univariate polynomials such that  $\varphi_i(v_j) = 0$  for  $i \neq j$  and  $\varphi_i(v_j) = 1$  for  $i = j$ . Suppose  $v_t = f_c$ . Then, the polynomial

$$(3.13) \quad \rho := \sum_{i < t} g_i(x) (\varphi_i(f(x)))^2 + \sum_{i \geq t} (\varphi_i(f(x)))^2$$

satisfies Assumption 3.2.

We refer to §2 for the archimedean condition and the notation  $\text{IQ}(h, g)$  as in (2.1). The following is about the convergence of relaxations (3.8)-(3.9).

**Theorem 3.3.** *Suppose  $\mathcal{K}_c \neq \emptyset$  and Assumption 3.1 holds. If*

- i)  $\text{IQ}(c_{eq}, c_{in}) + \text{IQ}(\phi, \psi)$  is archimedean, **or**
- ii)  $\text{IQ}(c_{eq}, c_{in})$  is archimedean, **or**
- iii) Assumption 3.2 holds,

*then  $f_k = f'_k = f_c$  for all  $k$  sufficiently large. Therefore, if the minimum value  $f_{\min}$  of (1.1) is achieved at a critical point, then  $f_k = f'_k = f_{\min}$  for all  $k$  big enough if one of the conditions i)-iii) is satisfied.*

*Remark:* In Theorem 3.3, the conclusion holds if one of conditions i)-iii) is satisfied. The condition ii) is only about constraining polynomials of (1.1). It can be checked without usage of  $\phi, \psi$ . Also note that the condition ii) implies the condition i).

*Proof of Theorem 3.3.* Clearly, every point in the complex variety

$$\mathcal{K}_1 := \{x \in \mathbb{C}^n \mid c_{eq}(x) = 0, \phi(x) = 0\}$$



is a complex critical point. By Lemma 3.3 of [6], the objective  $f$  achieves finitely many real values on  $\mathcal{K}_c = \mathcal{K}_1 \cap \mathbb{R}^n$ , say, they are  $v_1 < \dots < v_N$ . Up to the shifting of a constant in  $f$ , we can further assume that  $f_c = 0$ . Clearly,  $f_c$  equals one of  $v_1, \dots, v_N$ , say  $v_t = f_c = 0$ .

**Case I:** Assume the set  $\text{IQ}(c_{eq}, c_{in}) + \text{IQ}(\phi, \psi)$  is archimedean. Let

$$I := \text{Ideal}(c_{eq}, \phi),$$

the critical ideal. Note that  $\mathcal{K}_1 = \mathcal{V}_{\mathbb{C}}(I)$ . The variety  $\mathcal{V}_{\mathbb{C}}(I)$  is a union of irreducible subvarieties, say,  $V_1, \dots, V_\ell$ . If  $V_i \cap \mathbb{R}^n \neq \emptyset$ , then  $f$  is a real constant on  $V_i$ , which equals one of  $v_1, \dots, v_N$ . This can be implied by Lemma 3.3 of [6] and Lemma 3.2 of [28]. Denote the subvarieties of  $\mathcal{V}_{\mathbb{C}}(I)$ :

$$T_i := \mathcal{K}_1 \cap \{f(x) = v_i\} \quad (i = t, \dots, N).$$

Let  $T_{t-1}$  be the union of irreducible subvarieties  $V_i$ , such that either  $V_i \cap \mathbb{R}^n = \emptyset$  or  $f \equiv v_j$  on  $V_i$  with  $v_j < v_t = f_c$ . Then, it holds that

$$\mathcal{V}_{\mathbb{C}}(I) = T_{t-1} \cup T_t \cup \dots \cup T_N.$$

By the primary decomposition of  $I$  [11, 39], there exist ideals  $I_{t-1}, I_t, \dots, I_N \subseteq \mathbb{R}[x]$  such that

$$I = I_{t-1} \cap I_t \cap \dots \cap I_N$$

and  $T_i = \mathcal{V}_{\mathbb{C}}(I_i)$  for all  $i = t-1, t, \dots, N$ . Denote the semialgebraic set

$$(3.14) \quad S := \{x \in \mathbb{R}^n \mid c_{in}(x) \geq 0, \psi(x) \geq 0\}.$$

For  $i = t-1$ , we have  $\mathcal{V}_{\mathbb{R}}(I_{t-1}) \cap S = \emptyset$ , because  $v_1, \dots, v_{t-1} < f_c$  and  $w_1, \dots, w_M$  are not real. By the Positivstellensatz [2, Corollary 4.1.8], there exists  $p_0 \in \text{Preord}(c_{in}, \psi)^1$  satisfying  $2 + p_0 \in I_{t-1}$ . Note that  $1 + p_0 > 0$  on  $\mathcal{V}_{\mathbb{R}}(I_{t-1}) \cap S$ . The set  $I_{t-1} + \text{Qmod}(c_{in}, \psi)$  is archimedean, because  $I \subseteq I_{t-1}$  and

$$\text{IQ}(c_{eq}, c_{in}) + \text{IQ}(\phi, \psi) \subseteq I_{t-1} + \text{Qmod}(c_{in}, \psi).$$

By Theorem 2.1, we have

$$p_1 := 1 + p_0 \in I_{t-1} + \text{Qmod}(c_{in}, \psi).$$

Then  $1 + p_1 \in I_{t-1}$ . There exists  $p_2 \in \text{Qmod}(c_{in}, \psi)$  such that

$$-1 \equiv p_1 \equiv p_2 \pmod{I_{t-1}}.$$

Since  $f = (f/4 + 1)^2 - 1 \cdot (f/4 - 1)^2$ , we have

$$f \equiv \sigma_{t-1} := \{(f/4 + 1)^2 + p_2(f/4 - 1)^2\} \pmod{I_{t-1}}.$$

So, when  $k$  is big enough, we have  $\sigma_{t-1} \in \text{Qmod}(c_{in}, \psi)_{2k}$ .

For  $i = t$ ,  $v_t = 0$  and  $f(x)$  vanishes on  $\mathcal{V}_{\mathbb{C}}(I_t)$ . By Hilbert's Strong Nullstellensatz [4], there exists an integer  $m_t > 0$  such that  $f^{m_t} \in I_t$ . Hence,

$$s_t(\epsilon) := \sqrt{\epsilon} (1 + \epsilon^{-1} f)^{1/2} \equiv \sqrt{\epsilon} \sum_{j=0}^{m_t-1} \binom{1/2}{j} \epsilon^{-j} f^j \pmod{I_t}.$$

Let  $\sigma_t(\epsilon) = s_t(\epsilon)^2$ . Then  $f + \epsilon - \sigma_t(\epsilon) \in I_t$  and

$$f + \epsilon - \sigma_t(\epsilon) = \sum_{j=0}^{m_t-2} b_j(\epsilon) f^{m_t+j}$$

---

<sup>1</sup>It is the preordering of  $(c_{in}, \psi)$ ; see §7.1.

for some real numbers  $b_j(\epsilon)$ . Note that  $f^{m_t+j} \in I_t$  for all  $j \in \mathbb{N}$ .

For each  $i = t+1, \dots, N$ ,  $v_i > 0$  and  $f(x)/v_i - 1$  vanishes on  $\mathcal{V}_{\mathbb{C}}(I_i)$ . By Hilbert's Strong Nullstellensatz [4], there exists  $0 < m_i \in \mathbb{N}$  such that  $(f/v_i - 1)^{m_i} \in I_i$ . So,

$$\begin{aligned} s_i &:= \sqrt{v_i} \left( 1 + (f/v_i - 1) \right)^{1/2} \\ &\equiv \sqrt{v_i} \sum_{j=0}^{m_i-1} \binom{1/2}{j} (f/v_i - 1)^j \pmod{I_i}. \end{aligned}$$

Let  $\sigma_i = s_i^2$ , then  $f - \sigma_i \in I_i$ .

Note that  $\mathcal{V}_{\mathbb{C}}(I_i) \cap \mathcal{V}_{\mathbb{C}}(I_j) = \emptyset$  for all  $i \neq j$ . By Lemma 3.3 of [28], there exist  $a_{t-1}, \dots, a_N \in \mathbb{R}[x]$  such that

$$a_{t-1}^2 + \dots + a_N^2 - 1 \in I, \quad a_i \in \bigcap_{i \neq j \in \{t-1, \dots, N\}} I_j.$$

For  $\epsilon > 0$ , denote the polynomial

$$\sigma_\epsilon := \sigma_t(\epsilon) a_t^2 + \sum_{t \neq j \in \{t-1, \dots, N\}} (\sigma_j + \epsilon) a_j^2,$$

then

$$\begin{aligned} f + \epsilon - \sigma_\epsilon &= \sum_{t \neq i \in \{t-1, \dots, N\}} (f - \sigma_i) a_i^2 + (f + \epsilon - \sigma_t(\epsilon)) a_t^2 \\ &\quad + (f + \epsilon)(1 - a_{t-1}^2 - \dots - a_N^2). \end{aligned}$$

For each  $i \neq t$ ,  $f - \sigma_i \in I_i$ , so

$$(f - \sigma_i) a_i^2 \in \bigcap_{j=t-1}^N I_j = I.$$

Hence, there exists  $k_1 > 0$  such that

$$(f - \sigma_i) a_i^2 \in I_{2k_1} \quad (t \neq i \in \{t-1, \dots, N\}).$$

Since  $f + \epsilon - \sigma_t(\epsilon) \in I_t$ , we also have

$$(f + \epsilon - \sigma_t(\epsilon)) a_t^2 \in \bigcap_{j=t-1}^N I_j = I.$$

Moreover,

$$(f + \epsilon - \sigma_t(\epsilon)) a_t^2 = \sum_{j=0}^{m_t-2} b_j(\epsilon) f^{m_t+j} a_t^2.$$

Each  $f^{m_t+j} a_t^2 \in I$ , since  $f^{m_t+j} \in I_t$ . So, there exists  $k_2 > 0$  such that for all  $\epsilon > 0$

$$(f + \epsilon - \sigma_t(\epsilon)) a_t^2 \in I_{2k_2}.$$

Since  $1 - a_{t-1}^2 - \dots - a_N^2 \in I$ , there also exists  $k_3 > 0$  such that for all  $\epsilon > 0$

$$(f + \epsilon)(1 - a_{t-1}^2 - \dots - a_N^2) \in I_{2k_3}.$$

Hence, if  $k^* \geq \max\{k_1, k_2, k_3\}$ , then we have

$$f(x) + \epsilon - \sigma_\epsilon \in I_{2k^*}$$

for all  $\epsilon > 0$ . By the construction, the degrees of all  $\sigma_i$  and  $a_i$  are independent of  $\epsilon$ . So,  $\sigma_\epsilon \in \text{Qmod}(c_{in}, \psi)_{2k^*}$  for all  $\epsilon > 0$  if  $k^*$  is big enough. Note that

$$I_{2k^*} + \text{Qmod}(c_{in}, \psi)_{2k^*} = \text{IQ}(c_{eq}, c_{in})_{2k^*} + \text{IQ}(\phi, \psi)_{2k^*}.$$

This implies that  $f_{k^*} \geq f_c - \epsilon$  for all  $\epsilon > 0$ . On the other hand, we always have  $f_{k^*} \leq f_c$ . So,  $f_{k^*} = f_c$ . Moreover, since  $\{f_k\}$  is monotonically increasing, we must have  $f_k = f_c$  for all  $k \geq k^*$ .

**Case II:** Assume the set  $\text{IQ}(c_{eq}, c_{in})$  is archimedean. Because

$$\text{IQ}(c_{eq}, c_{in}) \subseteq \text{IQ}(c_{eq}, c_{in}) + \text{IQ}(\phi, \psi),$$

the set  $\text{IQ}(c_{eq}, c_{in}) + \text{IQ}(\phi, \psi)$  is also archimedean. Therefore, the conclusion is also true by applying the result for **Case I**.

**Case III:** Suppose the Assumption 3.2 holds. Let  $\varphi_1, \dots, \varphi_N$  be real univariate polynomials such that  $\varphi_i(v_j) = 0$  for  $i \neq j$  and  $\varphi_i(v_j) = 1$  for  $i = j$ . Let

$$s_i := (v_i - f_c)(\varphi_i(f))^2 \quad (i = 1, \dots, N)$$

and  $s := s_1 + \dots + s_N$ . Then,  $s \in \Sigma[x]_{2k_4}$  for some integer  $k_4 > 0$ . Let

$$\hat{f} := f - f_c - s.$$

We show that there exist an integer  $\ell > 0$  and  $q \in \text{Qmod}(c_{in}, \psi)$  such that

$$\hat{f}^{2\ell} + q \in \text{Ideal}(c_{eq}, \phi).$$

This is because, by Assumption 3.2,  $\hat{f}(x) \equiv 0$  on the set

$$\mathcal{K}_2 := \{x \in \mathbb{R}^n : c_{eq}(x) = 0, \phi(x) = 0, \rho(x) \geq 0\}.$$

It has only a single inequality. By the Positivstellensatz [2, Corollary 4.1.8], there exist  $0 < \ell \in \mathbb{N}$  and  $q = b_0 + \rho b_1$  ( $b_0, b_1 \in \Sigma[x]$ ) such that  $\hat{f}^{2\ell} + q \in \text{Ideal}(c_{eq}, \phi)$ . By Assumption 3.2,  $\rho \in \text{Qmod}(c_{in}, \psi)$ , so we have  $q \in \text{Qmod}(c_{in}, \psi)$ .

For all  $\epsilon > 0$  and  $\tau > 0$ , we have  $\hat{f} + \epsilon = \phi_\epsilon + \theta_\epsilon$  where

$$\phi_\epsilon = -\tau \epsilon^{1-2\ell} (\hat{f}^{2\ell} + q),$$

$$\theta_\epsilon = \epsilon \left( 1 + \hat{f}/\epsilon + \tau (\hat{f}/\epsilon)^{2\ell} \right) + \tau \epsilon^{1-2\ell} q.$$

By Lemma 2.1 of [29], when  $\tau \geq \frac{1}{2\ell}$ , there exists  $k_5$  such that, for all  $\epsilon > 0$ ,

$$\phi_\epsilon \in \text{Ideal}(c_{eq}, \phi)_{2k_5}, \quad \theta_\epsilon \in \text{Qmod}(c_{in}, \psi)_{2k_5}.$$

Hence, we can get

$$f - (f_c - \epsilon) = \phi_\epsilon + \sigma_\epsilon,$$

where  $\sigma_\epsilon = \theta_\epsilon + s \in \text{Qmod}(c_{in}, \psi)_{2k_5}$  for all  $\epsilon > 0$ . Note that

$$\text{IQ}(c_{eq}, c_{in})_{2k_5} + \text{IQ}(\phi, \psi)_{2k_5} = \text{Ideal}(c_{eq}, \phi)_{2k_5} + \text{Qmod}(c_{in}, \psi)_{2k_5}.$$

For all  $\epsilon > 0$ ,  $\gamma = f_c - \epsilon$  is feasible in (3.9) for the order  $k_5$ , so  $f_{k_5} \geq f_c$ . Because of (3.11) and the monotonicity of  $\{f_k\}$ , we have  $f_k = f'_k = f_c$  for all  $k \geq k_5$ .  $\square$

**3.2. Detecting tightness and extracting minimizers.** The optimal value of (3.7) is  $f_c$ , and the optimal value of (1.1) is  $f_{\min}$ . If  $f_{\min}$  is achievable at a critical point, then  $f_c = f_{\min}$ . In Theorem 3.3, we have shown that  $f_k = f_c$  for all  $k$  big enough, where  $f_k$  is the optimal value of (3.9). The value  $f_c$  or  $f_{\min}$  is often not known. How do we detect the tightness  $f_k = f_c$  in computation? The flat extension or flat truncation condition [5, 14, 30] can be used for checking tightness. Suppose  $y^*$  is a minimizer of (3.8) for the order  $k$ . Let

$$(3.15) \quad d := \lceil \deg(c_{eq}, c_{in}, \phi, \psi)/2 \rceil.$$

If there exists an integer  $t \in [d, k]$  such that

$$(3.16) \quad \text{rank } M_t(y^*) = \text{rank } M_{t-d}(y^*)$$

then  $f_k = f_c$  and we can get  $r := \text{rank } M_t(y^*)$  minimizers for (3.7) [5, 14, 30]. The method in [14] can be used to extract minimizers. It was implemented in the software **GloptiPoly 3** [13]. Generally, (3.16) can serve as a sufficient and necessary condition for detecting tightness. The case that (3.7) is infeasible (i.e., no critical points satisfy the constraints  $c_{in} \geq 0, \psi \geq 0$ ) can also be detected by solving the relaxations (3.8)-(3.9).

**Theorem 3.4.** *Under Assumption 3.1, the relaxations (3.8)-(3.9) have the following properties:*

- i) *If (3.8) is infeasible for some order  $k$ , then no critical points satisfy the constraints  $c_{in} \geq 0, \psi \geq 0$ , i.e., (3.7) is infeasible.*
- ii) *Suppose Assumption 3.2 holds. If (3.7) is infeasible, then the relaxation (3.8) must be infeasible when the order  $k$  is big enough.*

*In the following, assume (3.7) is feasible (i.e.,  $f_c < +\infty$ ). Then, for all  $k$  big enough, (3.8) has a minimizer  $y^*$ . Moreover,*

- iii) *If (3.16) is satisfied for some  $t \in [d, k]$ , then  $f_k = f_c$ .*
- iv) *If Assumption 3.2 holds and (3.7) has finitely many minimizers, then every minimizer  $y^*$  of (3.8) must satisfy (3.16) for some  $t \in [d, k]$ , when  $k$  is big enough.*

*Proof.* By Assumption 3.1,  $u$  is a critical point if and only if  $c_{eq}(u) = 0, \phi(u) = 0$ .

i) For every feasible point  $u$  of (3.7), the tms  $[u]_{2k}$  (see §2 for the notation) is feasible for (3.8), for all  $k$ . Therefore, if (3.8) is infeasible for some  $k$ , then (3.7) must be infeasible.

ii) By Assumption 3.2, when (3.7) is infeasible, the set

$$\{x \in \mathbb{R}^n : c_{eq}(x) = 0, \phi(x) = 0, \rho(x) \geq 0\}$$

is empty. It has a single inequality. By the Positivstellensatz (cf. [2, Corollary 4.1.8]), it holds that  $-1 \in \text{Ideal}(c_{eq}, \phi) + \text{Qmod}(\rho)$ . By Assumption 3.2,

$$\text{Ideal}(c_{eq}, \phi) + \text{Qmod}(\rho) \subseteq \text{IQ}(c_{eq}, c_{in}) + \text{IQ}(\phi, \psi).$$

Thus, for all  $k$  big enough, (3.9) is unbounded from above. Hence, (3.8) must be infeasible, by weak duality.

When (3.7) is feasible,  $f$  achieves finitely many values on  $\mathcal{K}_c$ , so (3.7) must achieve its optimal value  $f_c$ . By Theorem 3.3, we know that  $f_k = f'_k = f_c$  for all  $k$  big enough. For each minimizer  $u^*$  of (3.7), the tms  $[u^*]_{2k}$  is a minimizer of (3.8).

iii) If (3.16) holds, we can get  $r := \text{rank } M_t(y^*)$  minimizers for (3.7) (cf. [5, 14]), say,  $u_1, \dots, u_r$ , such that  $f_k = f(u_i)$  for each  $i$ . Clearly,  $f_k = f(u_i) \geq f_c$ . On the other hand, we always have  $f_k \leq f_c$ . So,  $f_k = f_c$ .

iv) By Assumption 3.2, (3.7) is equivalent to the problem

$$(3.17) \quad \begin{cases} \min & f(x) \\ \text{s.t.} & c_{eq}(x) = 0, \phi(x) = 0, \rho(x) \geq 0. \end{cases}$$

The optimal value of (3.17) is also  $f_c$ . Its  $k$ th order Lasserre's relaxation is

$$(3.18) \quad \begin{cases} \gamma'_k := \min & \langle f, y \rangle \\ \text{s.t.} & \langle 1, y \rangle = 1, M_k(y) \succeq 0, \\ & L_{c_{eq}}^{(k)}(y) = 0, L_{\phi}^{(k)}(y) = 0, L_{\rho}^{(k)}(y) \succeq 0. \end{cases}$$

The dual optimization problem is

$$(3.19) \quad \begin{cases} \gamma_k := \max & \gamma \\ \text{s.t.} & f - \gamma \in \text{Ideal}(c_{eq}, \phi)_{2k} + \text{Qmod}(\rho)_{2k}. \end{cases}$$

By repeating the same proof as for Theorem 3.3(iii), we can show that

$$\gamma_k = \gamma'_k = f_c$$

for all  $k$  big enough. Because  $\rho \in \text{Qmod}(c_{in}, \psi)$ , each  $y$  feasible for (3.8) is also feasible for (3.18). So, when  $k$  is big, each  $y^*$  is also a minimizer of (3.18). The problem (3.17) also has finitely many minimizers. By Theorem 2.6 of [30], the condition (3.16) must be satisfied for some  $t \in [d, k]$ , when  $k$  is big enough.  $\square$

If (3.7) has infinitely many minimizers, then the condition (3.16) is typically not satisfied. We refer to [24, §6.6].

#### 4. POLYHEDRAL CONSTRAINTS

In this section, we assume the feasible set of (1.1) is the polyhedron

$$P := \{x \in \mathbb{R}^n \mid Ax - b \geq 0\},$$

where  $A = [a_1 \ \dots \ a_m]^T \in \mathbb{R}^{m \times n}$ ,  $b = [b_1 \ \dots \ b_m]^T \in \mathbb{R}^m$ . This corresponds to that  $\mathcal{E} = \emptyset$ ,  $\mathcal{I} = [m]$ , and each  $c_i(x) = a_i^T x - b_i$ . Denote

$$(4.1) \quad D(x) := \text{diag}(c_1(x), \dots, c_m(x)), \quad C(x) := \begin{bmatrix} A^T \\ D(x) \end{bmatrix}.$$

The Lagrange multiplier vector  $\lambda := [\lambda_1 \ \dots \ \lambda_m]^T$  satisfies

$$(4.2) \quad \begin{bmatrix} A^T \\ D(x) \end{bmatrix} \lambda = \begin{bmatrix} \nabla f(x) \\ 0 \end{bmatrix}.$$

If  $\text{rank } A = m$ , we can express  $\lambda$  as

$$(4.3) \quad \lambda = (AA^T)^{-1} A \nabla f(x).$$

If  $\text{rank } A < m$ , how can we express  $\lambda$  in terms of  $x$ ? In computation, we often prefer a polynomial expression. If there exists  $L(x) \in \mathbb{R}[x]^{m \times (n+m)}$  such that

$$(4.4) \quad L(x)C(x) = I_m,$$

then we can get

$$\lambda = L(x) \begin{bmatrix} \nabla f(x) \\ 0 \end{bmatrix} = L_1(x) \nabla f(x),$$

where  $L_1(x)$  consists of the first  $n$  columns of  $L(x)$ . In this section, we characterize when such  $L(x)$  exists and give a degree bound for it.

We say that the linear function  $Ax - b$  is *nonsingular* if  $\text{rank } C(u) = m$  for all  $u \in \mathbb{C}^n$ . This is equivalent to that for every  $u$ , if  $J(u) = \{i_1, \dots, i_k\}$  (see (1.2) for the notation), then  $a_{i_1}, \dots, a_{i_k}$  are linearly independent.

**Proposition 4.1.** *The linear function  $Ax - b$  is nonsingular if and only if there exists a matrix polynomial  $L(x)$  satisfying (4.4). Moreover, when  $Ax - b$  is nonsingular, we can choose  $L(x)$  in (4.4) to have degree  $\leq m - \text{rank } A$ .*

*Proof.* Clearly, if (4.4) is satisfied by some  $L(x)$ , then  $\text{rank } C(u) \geq m$  for all  $u$ . This implies that  $Ax - b$  is nonsingular.

Next, assume that  $Ax - b$  is nonsingular. We show that (4.4) is satisfied by some  $L(x) \in \mathbb{R}[x]^{m \times (n+m)}$  with degree  $\leq m - \text{rank } A$ . Let  $r = \text{rank } A$ . Up to a linear coordinate transformation, we can reduce  $x$  to a  $r$ -dimensional variable. Without loss of generality, we can assume that  $\text{rank } A = n$  and  $m \geq n$ .

For a subset  $I := \{i_1, \dots, i_{m-n}\}$  of  $[m]$ , denote

$$c_I(x) := \prod_{i \in I} c_i(x), \quad E_I(x) := c_I(x) \cdot \text{diag}(c_{i_1}(x)^{-1}, \dots, c_{i_{m-n}}(x)^{-1}),$$

$$D_I(x) := \text{diag}(c_{i_1}(x), \dots, c_{i_{m-n}}(x)), \quad A_I = [a_{i_1} \ \cdots \ a_{i_{m-n}}]^T.$$

For the case that  $I = \emptyset$  (the empty set), we set  $c_\emptyset(x) = 1$ , by convention. Let

$$V = \{I \subseteq [m] : |I| = m - n, \text{rank } A_{[m] \setminus I} = n\}.$$

**Step I:** For each  $I \in V$ , we construct a matrix polynomial  $L_I(x)$  such that

$$(4.5) \quad L_I(x)C(x) = c_I(x)I_m.$$

The matrix  $L_I := L_I(x)$  satisfying (4.5) can be given by the following  $2 \times 3$  block matrix ( $L_I(\mathcal{J}, \mathcal{K})$  denotes the submatrix whose row indices are from  $\mathcal{J}$  and column indices from  $\mathcal{K}$ ):

$$(4.6) \quad \left( \begin{array}{c|cc} \mathcal{J} \setminus \mathcal{K} & [n] & n+I & n+[m] \setminus I \\ \hline I & 0 & E_I(x) & 0 \\ [m] \setminus I & c_I(x) \cdot (A_{[m] \setminus I})^{-T} & -(A_{[m] \setminus I})^{-T} (A_I)^T E_I(x) & 0 \end{array} \right).$$

Equivalently, the blocks of  $L_I$  are:

$$L_I(I, [n]) = 0, \quad L_I(I, n+[m] \setminus I) = 0, \quad L_I([m] \setminus I, n+[m] \setminus I) = 0,$$

$$L_I(I, n+I) = E_I(x), \quad L_I([m] \setminus I, [n]) = c_I(x)(A_{[m] \setminus I})^{-T},$$

$$L_I([m] \setminus I, n+I) = -(A_{[m] \setminus I})^{-T} (A_I)^T.$$

For each  $I \in V$ ,  $A_{[m] \setminus I}$  is invertible. The superscript  $^{-T}$  denotes the inverse of the transpose. Let  $G := L_I(x)C(x)$ , then one can verify that

$$G(I, I) = E_I(x)D_I(x) = c_I(x)I_{m-n}, \quad G(I, [m] \setminus I) = 0,$$

$$G([m] \setminus I, [m] \setminus I) = \begin{bmatrix} c_I(x)(A_{[m] \setminus I})^{-T} & -A_{[m] \setminus I}^{-T} A_I^T E_I(x) \end{bmatrix} \begin{bmatrix} (A_{[m] \setminus I})^T \\ 0 \end{bmatrix} = c_I(x)I_n.$$

$$G([m] \setminus I, I) = \begin{bmatrix} c_I(x)(A_{[m] \setminus I})^{-T} & -A_{[m] \setminus I}^{-T} (A_I)^T E_I(x) \end{bmatrix} \begin{bmatrix} A_I^T \\ D_I(x) \end{bmatrix} = 0.$$

This shows that the above  $L_I(x)$  satisfies (4.5).

**Step II:** We show that there exist real scalars  $\nu_I$  satisfying

$$(4.7) \quad \sum_{I \in V} \nu_I c_I(x) = 1.$$

This can be shown by induction on  $m$ .

- When  $m = n$ ,  $V = \emptyset$  and  $c_\emptyset(x) = 1$ , so (4.7) is clearly true.
- When  $m > n$ , let

$$(4.8) \quad N := \{i \in [m] \mid \text{rank } A_{[m] \setminus \{i\}} = n\}.$$

For each  $i \in N$ , let  $V_i$  be the set of all  $I' \subseteq [m] \setminus \{i\}$  such that  $|I'| = m - n - 1$  and  $\text{rank } A_{[m] \setminus (I' \cup \{i\})} = n$ . For each  $i \in N$ , by the assumption, the linear function  $A_{m \setminus \{i\}}x - b_{m \setminus \{i\}}$  is nonsingular. By induction, there exist real scalars  $\nu_{I'}^{(i)}$  satisfying

$$(4.9) \quad \sum_{I' \in V_i} \nu_{I'}^{(i)} c_{I'}(x) = 1.$$

Since  $\text{rank } A = n$ , we can generally assume that  $\{a_1, \dots, a_n\}$  is linearly independent. So, there exist scalars  $\alpha_1, \dots, \alpha_n$  such that

$$a_m = \alpha_1 a_1 + \dots + \alpha_n a_n.$$

If all  $\alpha_i = 0$ , then  $a_m = 0$ . Then  $A$  can be replaced by its first  $m - 1$  rows. So, (4.7) is true by induction.

Now, suppose at least one  $\alpha_i \neq 0$  and

$$\{i : \alpha_i \neq 0\} = \{i_1, \dots, i_k\}.$$

Then  $a_{i_1}, \dots, a_{i_k}, a_m$  are linearly dependent. For convenience, set  $i_{k+1} := m$ . Since  $Ax - b$  is nonsingular, the linear system

$$c_{i_1}(x) = \dots = c_{i_k}(x) = c_{i_{k+1}}(x) = 0$$

has no solutions. Hence, there exist real scalars  $\mu_1, \dots, \mu_{k+1}$  such that

$$\mu_1 c_{i_1}(x) + \dots + \mu_k c_{i_k}(x) + \mu_{k+1} c_{i_{k+1}}(x) = 1.$$

This above can be implied by echelon's form for inconsistent linear systems. Note that  $i_1, \dots, i_{k+1} \in N$ . For each  $j = 1, \dots, k + 1$ , by (4.9),

$$\sum_{I' \in V_{i_j}} \nu_{I'}^{(i_j)} c_{I'}(x) = 1.$$

Then, we can get

$$\begin{aligned} 1 &= \sum_{j=1}^{k+1} \mu_j c_{i_j}(x) = \sum_{j=1}^{k+1} \mu_j \sum_{I' \in V_{i_j}} \nu_{I'}^{(i_j)} c_{i_j}(x) c_{I'}(x) = \\ &\quad \sum_{I = I' \cup \{i_j\}, I' \in V_{i_j}, 1 \leq j \leq k+1} \nu_{I'}^{(i_j)} \mu_j c_I(x). \end{aligned}$$

Since each  $I' \cup \{i_j\} \in V$ , (4.7) must be satisfied by some scalars  $\nu_I$ .

**Step III:** For  $L_I(x)$  as in (4.5), we construct  $L(x)$  as

$$(4.10) \quad L(x) := \sum_{I \in V} \nu_I c_I(x) L_I(x).$$

Clearly,  $L(x)$  satisfies (4.4) because

$$L(x)C(x) = \sum_{I \in V} \nu_I L_I(x)C(x) = \sum_{I \in V} \nu_I c_I(x) I_m = I_m.$$

Each  $L_I(x)$  has degree  $\leq m - n$ , so  $L(x)$  has degree  $\leq m - n$ .  $\square$

Proposition 4.1 characterizes when there exists  $L(x)$  satisfying (4.4). When it does, a degree bound for  $L(x)$  is  $m - \text{rank } A$ . Sometimes, its degree can be smaller than that, as shown in Example 4.3. For given  $A, b$ , the matrix polynomial  $L(x)$  satisfying (4.4) can be determined by linear equations, which are obtained by matching coefficients on both sides. In the following, we give some examples of  $L(x)C(x) = I_m$  for polyhedral sets.

**Example 4.2.** Consider the simplicial set

$$x_1 \geq 0, \dots, x_n \geq 0, 1 - e^T x \geq 0.$$

The equation  $L(x)C(x) = I_{n+1}$  is satisfied by

$$L(x) = \begin{bmatrix} 1 - x_1 & -x_2 & \cdots & -x_n & 1 & \cdots & 1 \\ -x_1 & 1 - x_2 & \cdots & -x_n & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ -x_1 & -x_2 & \cdots & 1 - x_n & 1 & \cdots & 1 \\ -x_1 & -x_2 & \cdots & -x_n & 1 & \cdots & 1 \end{bmatrix}.$$

**Example 4.3.** Consider the box constraint

$$x_1 \geq 0, \dots, x_n \geq 0, 1 - x_1 \geq 0, \dots, 1 - x_n \geq 0.$$

The equation  $L(x)C(x) = I_{2n}$  is satisfied by

$$L(x) = \begin{bmatrix} I_n - \text{diag}(x) & I_n & I_n \\ -\text{diag}(x) & I_n & I_n \end{bmatrix}.$$

**Example 4.4.** Consider the polyhedral set

$$1 - x_4 \geq 0, x_4 - x_3 \geq 0, x_3 - x_2 \geq 0, x_2 - x_1 \geq 0, x_1 + 1 \geq 0.$$

The equation  $L(x)C(x) = I_5$  is satisfied by

$$L(x) = \frac{1}{2} \begin{bmatrix} -x_1 - 1 & -x_2 - 1 & -x_3 - 1 & -x_4 - 1 & 1 & 1 & 1 & 1 & 1 \\ -x_1 - 1 & -x_2 - 1 & -x_3 - 1 & 1 - x_4 & 1 & 1 & 1 & 1 & 1 \\ -x_1 - 1 & -x_2 - 1 & 1 - x_3 & 1 - x_4 & 1 & 1 & 1 & 1 & 1 \\ -x_1 - 1 & 1 - x_2 & 1 - x_3 & 1 - x_4 & 1 & 1 & 1 & 1 & 1 \\ 1 - x_1 & 1 - x_2 & 1 - x_3 & 1 - x_4 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

**Example 4.5.** Consider the polyhedral set

$$1 + x_1 \geq 0, 1 - x_1 \geq 0, 2 - x_1 - x_2 \geq 0, 2 - x_1 + x_2 \geq 0.$$

The matrix  $L(x)$  satisfying  $L(x)C(x) = I_4$  is

$$\frac{1}{6} \begin{bmatrix} x_1^2 - 3x_1 + 2 & x_1 x_2 - x_2 & 4 - x_1 & 2 - x_1 & 1 - x_1 & 1 - x_1 \\ 3x_1^2 - 3x_1 - 6 & 3x_2 + 3x_1 x_2 & 6 - 3x_1 & -3x_1 & -3x_1 - 3 & -3x_1 - 3 \\ 1 - x_1^2 & -2x_2 - x_1 x_2 - 3 & x_1 - 1 & x_1 + 1 & x_1 + 2 & x_1 + 2 \\ 1 - x_1^2 & 3 - x_1 x_2 - 2x_2 & x_1 - 1 & x_1 + 1 & x_1 + 2 & x_1 + 2 \end{bmatrix}.$$



## 5. GENERAL CONSTRAINTS

We consider general constraints as in (1.1). The critical point conditions are in (1.8). We discuss how to express Lagrange multipliers  $\lambda_i$  as polynomial functions in  $x$  over the set of critical points.

Suppose there are totally  $m$  equality and inequality constraints, i.e.,

$$\mathcal{E} \cup \mathcal{I} = \{1, \dots, m\}.$$

If  $(x, \lambda)$  is a critical pair, then  $\lambda_i c_i(x) = 0$  for all  $i \in \mathcal{E} \cup \mathcal{I}$ . So, the vector  $\lambda := (\lambda_1, \dots, \lambda_m)^T$  of Lagrange multipliers satisfies the equation

$$(5.1) \quad \underbrace{\begin{bmatrix} \nabla c_1(x) & \nabla c_2(x) & \cdots & \nabla c_m(x) \\ c_1(x) & 0 & \cdots & 0 \\ 0 & c_2(x) & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & c_m(x) \end{bmatrix}}_{C(x)} \lambda = \begin{bmatrix} \nabla f(x) \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Let  $C(x)$  be as in (5.1). If there exists  $L(x) \in \mathbb{R}[x]^{m \times (m+n)}$  such that

$$(5.2) \quad L(x)C(x) = I_m,$$

then we can get

$$(5.3) \quad \lambda = L(x) \begin{bmatrix} \nabla f(x) \\ 0 \end{bmatrix} = L_1(x) \nabla f(x),$$

where  $L_1(x)$  consists of the first  $n$  columns of  $L(x)$ . This section characterizes when such  $L(x)$  exists.

Let  $c := (c_1, \dots, c_m)$  be the tuple of constraining polynomials. We say that  $c$  is nonsingular if  $\text{rank } C(u) = m$  for every  $u \in \mathbb{C}^n$ . This is equivalent to that for each  $u \in \mathbb{C}^n$ , if  $J(u) = \{i_1, \dots, i_k\}$  (see (1.2) for the notation), then the gradients  $\nabla c_{i_1}(u), \dots, \nabla c_{i_k}(u)$  are linearly independent. Our main conclusion is that (5.2) holds if and only if the tuple  $c$  is nonsingular.

**Proposition 5.1.** (i) For each  $W(x) \in \mathbb{C}[x]^{s \times t}$  with  $s \geq t$ ,  $\text{rank } W(u) = t$  for all  $u \in \mathbb{C}^n$  if and only if there exists  $P(x) \in \mathbb{C}[x]^{t \times s}$  such that

$$P(x)W(x) = I_t.$$

Moreover, for  $W(x) \in \mathbb{R}[x]^{s \times t}$ , we can choose  $P(x) \in \mathbb{R}[x]^{t \times s}$  for the above.

(ii) The constraining polynomial tuple  $c$  is nonsingular if and only if there exists  $L(x) \in \mathbb{R}[x]^{m \times (m+n)}$  satisfying (5.2).

*Proof.* (i) “ $\Leftarrow$ ”: If  $L(x)W(x) = I_t$ , then for all  $u \in \mathbb{C}^n$

$$t = \text{rank } I_t \leq \text{rank } W(u) \leq t.$$

So,  $W(x)$  must have full column rank everywhere.

“ $\Rightarrow$ ”: Suppose  $\text{rank } W(u) = t$  for all  $u \in \mathbb{C}^n$ . Write  $W(x)$  in the column form

$$W(x) = \begin{bmatrix} w_1(x) & w_2(x) & \cdots & w_t(x) \end{bmatrix}.$$

Then, the equation  $w_1(x) = 0$  does not have a complex solution. By Hilbert's Weak Nullstellensatz [4], there exists  $\xi_1(x) \in \mathbb{C}[x]^s$  such that  $\xi_1(x)^T w_1(x) = 1$ . For each  $i = 2, \dots, t$ , denote

$$r_{1,i}(x) := \xi_1(x)^T w_i(x),$$

then (use  $\sim$  to denote row equivalence between matrices)

$$W(x) \sim \begin{bmatrix} 1 & r_{1,2}(x) & \cdots & r_{1,t}(x) \\ w_1(x) & w_2(x) & \cdots & w_m(x) \end{bmatrix} \sim W_1(x) := \begin{bmatrix} 1 & r_{1,2}(x) & \cdots & r_{1,m}(x) \\ 0 & w_2^{(1)}(x) & \cdots & w_m^{(1)}(x) \end{bmatrix},$$

where each  $(i = 2, \dots, m)$

$$w_i^{(1)}(x) = w_i(x) - r_{1,i}(x)w_1(x).$$

So, there exists  $P_1(x) \in \mathbb{R}[x]^{(s+1) \times s}$  such that

$$P_1(x)W(x) = W_1(x).$$

Since  $W(x)$  and  $W_1(x)$  are row equivalent,  $W_1(x)$  must also have full column rank everywhere. Similarly, the polynomial equation

$$w_2^{(1)}(x) = 0$$

does not have a complex solution. Again, by Hilbert's Weak Nullstellensatz [4], there exists  $\xi_2(x) \in \mathbb{C}[x]^s$  such that

$$\xi_2(x)^T w_2^{(1)}(x) = 1.$$

For each  $i = 3, \dots, t$ , let  $r_{2,i}(x) := \xi_2(x)^T w_i^{(1)}(x)$ , then

$$\begin{aligned} W_1(x) &\sim \begin{bmatrix} 1 & r_{1,2}(x) & r_{1,3}(x) & \cdots & r_{1,m}(x) \\ 0 & 1 & r_{2,3}(x) & \cdots & r_{2,m}(x) \\ 0 & w_2^{(1)}(x) & w_3^{(1)}(x) & \cdots & w_m^{(1)}(x) \end{bmatrix} \sim \\ W_2(x) &:= \begin{bmatrix} 1 & r_{1,2}(x) & r_{1,3}(x) & \cdots & r_{1,m}(x) \\ 0 & 1 & r_{2,3}(x) & \cdots & r_{2,m}(x) \\ 0 & 0 & w_3^{(2)}(x) & \cdots & w_m^{(2)}(x) \end{bmatrix}, \end{aligned}$$

where each  $(i = 3, \dots, m)$

$$w_i^{(2)}(x) = w_i^{(1)}(x) - r_{2,i}(x)w_2^{(1)}(x).$$

Similarly,  $W_1(x)$  and  $W_2(x)$  are row equivalent, so  $W_2(x)$  has full column rank everywhere. There exists  $P_2(x) \in \mathbb{C}[x]^{(s+2) \times (s+1)}$  such that

$$P_2(x)W_1(x) = W_2(x).$$

Continuing this process, we can finally get

$$W_2(x) \sim \cdots \sim W_t(x) := \begin{bmatrix} 1 & r_{1,2}(x) & r_{1,3}(x) & \cdots & r_{1,t}(x) \\ 0 & 1 & r_{2,3}(x) & \cdots & r_{2,t}(x) \\ 0 & 0 & 1 & \cdots & r_{3,t}(x) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Consequently, there exists  $P_i(x) \in \mathbb{R}[x]^{(s+i) \times (s+i-1)}$  for  $i = 1, 2, \dots, t$ , such that

$$P_t(x)P_{t-1}(x) \cdots P_1(x)W(x) = W_t(x).$$

Since  $W_t(x)$  is a unit upper triangular matrix polynomial, there exists  $P_{t+1}(x) \in \mathbb{R}[x]^{t \times (s+t)}$  such that  $P_{t+1}(x)W_t(x) = I_t$ . Let

$$P(x) := P_{t+1}(x)P_t(x)P_{t-1}(x) \cdots P_1(x),$$

then  $P(x)W(x) = I_m$ . Note that  $P(x) \in \mathbb{C}[x]^{t \times s}$ .

For  $W(x) \in \mathbb{R}[x]^{s \times t}$ , we can replace  $P(x)$  by  $(P(x) + \overline{P(x)})/2$  (the  $\overline{P(x)}$  denotes the complex conjugate of  $P(x)$ ), which is a real matrix polynomial.

(ii) The conclusion is implied directly by the item (i).  $\square$

In Proposition 5.1, we do not have a degree bound for  $L(x)$  satisfying (5.2). This question is mostly open, to the best of the author's knowledge. However, once a degree is chosen for  $L(x)$ , it can be determined by comparing coefficients of both sides of (5.2). This can be done by solving a linear system. In the following, we give some examples of  $L(x)$  satisfying (5.2).

**Example 5.2.** Consider the hypercube with quadratic constraints

$$1 - x_1^2 \geq 0, 1 - x_2^2 \geq 0, \dots, 1 - x_n^2 \geq 0.$$

The equation  $L(x)C(x) = I_n$  is satisfied by

$$L(x) = \begin{bmatrix} -\frac{1}{2}\text{diag}(x) & I_n \end{bmatrix}.$$

**Example 5.3.** Consider the nonnegative portion of the unit sphere

$$x_1 \geq 0, x_2 \geq 0, \dots, x_n \geq 0, x_1^2 + \dots + x_n^2 - 1 = 0.$$

The equation  $L(x)C(x) = I_{n+1}$  is satisfied by

$$L(x) = \begin{bmatrix} I_n - xx^T & x\mathbf{1}_n^T & 2x \\ \frac{1}{2}x^T & -\frac{1}{2}\mathbf{1}_n^T & -1 \end{bmatrix}.$$

**Example 5.4.** Consider the set

$$1 - x_1^3 - x_2^4 \geq 0, 1 - x_3^4 - x_4^3 \geq 0.$$

The equation  $L(x)C(x) = I_2$  is satisfied by

$$L(x) = \begin{bmatrix} -\frac{x_1}{3} & -\frac{x_2}{4} & 0 & 0 & 1 & 0 \\ 0 & 0 & -\frac{x_3}{4} & -\frac{x_4}{3} & 0 & 1 \end{bmatrix}.$$

**Example 5.5.** Consider the quadratic set

$$1 - x_1x_2 - x_2x_3 - x_1x_3 \geq 0, 1 - x_1^2 - x_2^2 - x_3^2 \geq 0.$$

The matrix  $L(x)^T$  satisfying  $L(x)C(x) = I_2$  is

$$\begin{bmatrix} 25x_1^3 + 10x_1^2x_2 + 40x_1x_2^2 - 25x_1 - 2x_3 & -25x_1^3 - 10x_1^2x_2 - 40x_1x_2^2 + \frac{49x_1}{2} + 2x_3 \\ -15x_1^2x_2 + 10x_1x_2^2 + 20x_3x_1x_2 - 10x_1 & 15x_1^2x_2 - 10x_1x_2^2 - 20x_3x_1x_2 + 10x_1 - \frac{x_2}{2} \\ 25x_3x_1^2 - 20x_1x_2^2 + 10x_3x_1x_2 + 2x_1 & -25x_3x_1^2 + 20x_1x_2^2 - 10x_3x_1x_2 - 2x_1 - \frac{x_3}{2} \\ 1 - 20x_1x_3 - 10x_1^2 - 20x_1x_2 & 20x_1x_2 + 20x_1x_3 + 10x_1^2 \\ -50x_1^2 - 20x_2x_1 & 50x_1^2 + 20x_2x_1 + 1 \end{bmatrix}.$$

## 6. NUMERICAL EXAMPLES

This section gives examples of using the new relaxations (3.8)-(3.9) for solving the optimization problem (1.1), with usage of Lagrange multiplier expressions. Some polynomials in the examples are from [35]. The computation is implemented in MATLAB R2012a, on a Lenovo Laptop with CPU@2.90GHz and RAM 16.0G. The relaxations (3.8)-(3.9) are solved by the software `GloptiPoly 3` [13], which calls the semidefinite program solver `SeDuMi` [38]. For neatness, only four decimal digits are displayed for computational results.

The polynomials  $p_i$  in Assumption 3.1 are constructed as follows. Order the constraining polynomials as  $c_1, \dots, c_m$ . First, find a matrix polynomial  $L(x)$  satisfying (4.4) or (5.2). Let  $L_1(x)$  be the submatrix of  $L(x)$ , consisting of the first  $n$  columns. Then, choose  $(p_1, \dots, p_m)$  to be the product  $L_1(x)\nabla f(x)$ , i.e.,

$$p_i = \left( L_1(x)\nabla f(x) \right)_i.$$

In our examples, the global minimum value  $f_{\min}$  of (1.1) is achieved at a critical point. This is the case if the feasible set is compact (or if  $f$  is coercive, i.e., for each  $\ell$ , the sublevel set  $\{f(x) \leq \ell\}$  is compact) and the constraint qualification condition holds. By Theorem 3.3, we can get  $f_k = f_{\min}$  for all  $k$  big enough, if one of the conditions i)-iii) there holds.

We compare the new relaxations (3.8)-(3.9) with standard Lasserre relaxations in [16]. The lower bounds given by relaxations in [16] (without using Lagrange multiplier expressions) and the lower bounds given by (3.8)-(3.9) (using Lagrange multiplier expressions) are shown in the tables. The computational time (in seconds) is also compared. The results for standard Lasserre relaxations are titled “Without L.M.E.”, and those for the new relaxations (3.8)-(3.9) are titled “With L.M.E.”. The new relaxations (3.8)-(3.9) not only give tighter lower bounds but also consume less time (except a few cases).

**Example 6.1.** Consider the optimization problem

$$\begin{cases} \min & x_1 x_2 (10 - x_3) \\ \text{s.t.} & x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, 1 - x_1 - x_2 - x_3 \geq 0. \end{cases}$$

The matrix polynomial  $L(x)$  is given in Example 4.2. Since the feasible set is compact, the minimum  $f_{\min} = 0$  is achieved at a critical point. Each feasible point  $(x_1, x_2, x_3)$  with  $x_1 x_2 = 0$  is a global minimizer. The computational results for standard Lasserre’s relaxations and the new ones (3.8)-(3.9) are in Table 1. It confirms that  $f_k = f_{\min}$  for all  $k \geq 3$ , up to numerical round-off errors.

TABLE 1. Computational results for Example 6.1.

order $k$	Without L.M.E.		With L.M.E.	
	lower bound	time	lower bound	time
2	−0.0521	0.6841	−0.0521	0.1922
3	−0.0026	0.2657	$-3 \cdot 10^{-8}$	0.2285
4	−0.0007	0.6785	$-6 \cdot 10^{-9}$	0.4431
5	−0.0004	1.6105	$-2 \cdot 10^{-9}$	0.9567

**Example 6.2.** Consider the optimization problem

$$\begin{cases} \min & x_1^4 x_2^2 + x_1^2 x_2^4 + x_3^6 - 3x_1^2 x_2^2 x_3^2 + (x_1^4 + x_2^4 + x_3^4) \\ \text{s.t.} & x_1^2 + x_2^2 + x_3^2 \geq 1. \end{cases}$$

The matrix polynomial  $L(x) = [\frac{1}{2}x_1 \quad \frac{1}{2}x_2 \quad \frac{1}{2}x_3 \quad -1]$ . The objective  $f$  is the sum of the Motzkin polynomial (nonnegative everywhere but not SOS [35]) and the positive definite form  $x_1^4 + x_2^4 + x_3^4$ . So,  $f$  is coercive and  $f_{\min}$  is achieved at a critical point. Clearly,  $f_{\min} = \frac{1}{3}$ . There are 8 minimizers  $(\pm \frac{1}{\sqrt{3}}, \pm \frac{1}{\sqrt{3}}, \pm \frac{1}{\sqrt{3}})$ . The computational results for standard Lasserre's relaxations and the new ones (3.8)-(3.9) are in Table 2. It confirms that  $f_k = f_{\min}$  for all  $k \geq 4$ , up to numerical round-off errors.

TABLE 2. Computational results for Example 6.2.

order $k$	Without L.M.E.		With L.M.E.	
	lower bound	time	lower bound	time
3	$-\infty$	0.4466	0.1111	0.1169
4	$-\infty$	0.4948	0.3333	0.3499
5	$-2.1821 \cdot 10^5$	1.1836	0.3333	0.6530

**Example 6.3.** Consider the optimization problem:

$$\begin{cases} \min & x_1 x_2 + x_2 x_3 + x_3 x_4 - 3x_1 x_2 x_3 x_4 + (x_1^3 + \dots + x_4^3) \\ \text{s.t.} & x_1, x_2, x_3, x_4 \geq 0, 1 - x_1 - x_2 \geq 0, 1 - x_3 - x_4 \geq 0. \end{cases}$$

The matrix polynomial  $L(x)$  is

$$\begin{bmatrix} 1-x_1 & -x_2 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ -x_1 & 1-x_2 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1-x_3 & -x_4 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & -x_3 & 1-x_4 & 0 & 0 & 1 & 1 & 0 & 1 \\ -x_1 & -x_2 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -x_3 & -x_4 & 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}.$$

The feasible set is compact, so  $f_{\min}$  is achieved at a critical point. One can show that  $f_{\min} = 0$  and the minimizer is the origin. The computational results for standard Lasserre's relaxations and the new ones (3.8)-(3.9) are in Table 3.

TABLE 3. Computational results for Example 6.3.

order $k$	Without L.M.E.		With L.M.E.	
	lower bound	time	lower bound	time
3	$-2.9 \cdot 10^{-5}$	0.7335	$-6 \cdot 10^{-7}$	0.6091
4	$-1.4 \cdot 10^{-5}$	2.5055	$-8 \cdot 10^{-8}$	2.7423
5	$-1.4 \cdot 10^{-5}$	12.7092	$-5 \cdot 10^{-8}$	13.7449

**Example 6.4.** Consider the polynomial optimization problem

$$\begin{cases} \min_{x \in \mathbb{R}^2} & x_1^2 + 50x_2^2 \\ \text{s.t.} & x_1^2 - \frac{1}{2} \geq 0, x_2^2 - 2x_1 x_2 - \frac{1}{8} \geq 0, x_2^2 + 2x_1 x_2 - \frac{1}{8} \geq 0. \end{cases}$$

It is motivated from an example in [12, §3]. The first column of  $L(x)$  is

$$\begin{bmatrix} \frac{8x_1^3}{5} + \frac{x_1}{5} \\ \frac{288x_2x_1^4}{5} - \frac{16x_1^3}{5} - \frac{x_2x_1^2124}{5} + \frac{8x_1}{5} - 2x_2 \\ -\frac{288x_2x_1^4}{5} - \frac{16x_1^3}{5} + \frac{x_2x_1^2124}{5} + \frac{8x_1}{5} + 2x_2 \end{bmatrix},$$

and the second column of  $L(x)$  is

$$\begin{bmatrix} -\frac{8x_1^2x_2}{5} + \frac{4x_2^3}{5} - \frac{x_2}{10} \\ \frac{288x_1^3x_2^2}{5} + \frac{16x_1^2x_2}{5} - \frac{142x_1x_2^2}{5} - \frac{9x_1}{20} - \frac{8x_2^3}{5} + \frac{11x_2}{5} \\ -\frac{288x_1^3x_2^2}{5} + \frac{16x_1^2x_2}{5} + \frac{142x_1x_2^2}{5} + \frac{9x_1}{20} - \frac{8x_2^3}{5} + \frac{11x_2}{5} \end{bmatrix}.$$

The objective is coercive, so  $f_{\min}$  is achieved at a critical point. The minimum value  $f_{\min} = 56 + 3/4 + 25\sqrt{5} \approx 112.6517$  and the minimizers are  $(\pm\sqrt{1/2}, \pm(\sqrt{5/8} + \sqrt{1/2}))$ . The computational results for standard Lasserre's relaxations and the new ones (3.8)-(3.9) are in Table 4. It confirms that  $f_k = f_{\min}$  for all  $k \geq 4$ , up to numerical round-off errors.

TABLE 4. Computational results for Example 6.4.

order $k$	Without L.M.E.		With L.M.E.	
	lower bound	time	lower bound	time
3	6.7535	0.4611	56.7500	0.1309
4	6.9294	0.2428	112.6517	0.2405
5	8.8519	0.3376	112.6517	0.2167
6	16.5971	0.4703	112.6517	0.3788
7	35.4756	0.6536	112.6517	0.4537

**Example 6.5.** Consider the optimization problem

$$\begin{cases} \min_{x \in \mathbb{R}^3} & x_1^3 + x_2^3 + x_3^3 + 4x_1x_2x_3 - (x_1(x_2^2 + x_3^2) + x_2(x_3^2 + x_1^2) + x_3(x_1^2 + x_2^2)) \\ \text{s.t.} & x_1 \geq 0, x_1x_2 - 1 \geq 0, x_2x_3 - 1 \geq 0. \end{cases}$$

The matrix polynomial  $L(x)$  is

$$\begin{bmatrix} 1 - x_1x_2 & 0 & 0 & x_2 & x_2 & 0 \\ & x_1 & 0 & 0 & -1 & -1 & 0 \\ & -x_1 & x_2 & 0 & 1 & 0 & -1 \end{bmatrix}.$$

The objective is a variation of Robinson's form [35]. It is a positive definite form over the nonnegative orthant  $\mathbb{R}_+^3$ . In computation, we got  $f_{\min} \approx 0.9492$  and a global minimizer  $(0.9071, 1.1024, 0.9071)$ . The computational results for standard Lasserre's relaxations and the new ones (3.8)-(3.9) are in Table 5. It confirms that  $f_k = f_{\min}$  for all  $k \geq 3$ , up to numerical round-off errors.

**Example 6.6.** Consider the optimization problem ( $x_0 := 1$ )

$$\begin{cases} \min_{x \in \mathbb{R}^4} & x^T x + \sum_{i=0}^4 \prod_{j \neq i} (x_i - x_j) \\ \text{s.t.} & x_1^2 - 1 \geq 0, x_2^2 - 1 \geq 0, x_3^2 - 1 \geq 0, x_4^2 - 1 \geq 0. \end{cases}$$

The matrix polynomial  $L(x) = [\frac{1}{2}\text{diag}(x) \quad -I_4]$ . The first part of the objective is  $x^T x$ , while the second part is a nonnegative polynomial [35]. The objective is

TABLE 5. Computational results for Example 6.5.

order $k$	Without L.M.E.		With L.M.E.	
	lower bound	time	lower bound	time
2	$-\infty$	0.4129	$-\infty$	0.1900
3	$-7.8184 \cdot 10^6$	0.4641	0.9492	0.3139
4	$-2.0575 \cdot 10^4$	0.6499	0.9492	0.5057

coercive, so  $f_{\min}$  is achieved at a critical point. In computation, we got  $f_{\min} = 4.0000$  and 11 global minimizers:

$$\begin{aligned}
&(1, 1, 1, 1), \quad (1, -1, -1, 1), \quad (1, -1, 1, -1), \quad (1, 1, -1, -1), \\
&(1, -1, -1, -1), \quad (-1, -1, 1, 1), \quad (-1, 1, -1, 1), \quad (-1, 1, 1, -1), \\
&(-1, -1, -1, 1), \quad (-1, -1, 1, -1), \quad (-1, 1, -1, -1).
\end{aligned}$$

The computational results for standard Lasserre's relaxations and the new ones (3.8)-(3.9) are in Table 6. It confirms that  $f_k = f_{\min}$  for all  $k \geq 4$ , up to numerical round-off errors.

TABLE 6. Computational results for Example 6.6.

order $k$	Without L.M.E.		With L.M.E.	
	lower bound	time	lower bound	time
3	$-\infty$	1.1377	3.5480	1.1765
4	$-6.6913 \cdot 10^4$	4.7677	4.0000	3.0761
5	-21.3778	22.9970	4.0000	10.3354

**Example 6.7.** Consider the optimization problem

$$\begin{cases} \min_{x \in \mathbb{R}^3} & x_1^4 x_2^2 + x_2^4 x_3^2 + x_3^4 x_1^2 - 3x_1^2 x_2^2 x_3^2 + x_2^2 \\ \text{s.t.} & x_1 - x_2 x_3 \geq 0, -x_2 + x_3^2 \geq 0. \end{cases}$$

The matrix polynomial  $L(x) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -x_3 & -1 & 0 & 0 & 0 \end{bmatrix}$ . By the arithmetic-geometric mean inequality, one can show that  $f_{\min} = 0$ . The global minimizers are  $(x_1, 0, x_3)$  with  $x_1 \geq 0$  and  $x_1 x_3 = 0$ . The computational results for standard Lasserre's relaxations and the new ones (3.8)-(3.9) are in Table 7. It confirms that  $f_k = f_{\min}$  for all  $k \geq 5$ , up to numerical round-off errors.

TABLE 7. Computational results for Example 6.7.

order $k$	Without L.M.E.		With L.M.E.	
	lower bound	time	lower bound	time
3	$-\infty$	0.6144	$-\infty$	0.3418
4	$-1.0909 \cdot 10^7$	1.0542	-3.9476	0.7180
5	-942.6772	1.6771	$-3 \cdot 10^{-9}$	1.4607
6	-0.0110	3.3532	$-8 \cdot 10^{-10}$	3.1618

**Example 6.8.** Consider the optimization problem

$$\begin{cases} \min_{x \in \mathbb{R}^4} & x_1^2(x_1 - x_4)^2 + x_2^2(x_2 - x_4)^2 + x_3^2(x_3 - x_4)^2 + \\ & 2x_1x_2x_3(x_1 + x_2 + x_3 - 2x_4) + (x_1 - 1)^2 + (x_2 - 1)^2 + (x_3 - 1)^2 \\ \text{s.t.} & x_1 - x_2 \geq 0, x_2 - x_3 \geq 0. \end{cases}$$

The matrix polynomial  $L(x) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$ . In the objective, the sum of the first 4 terms is a nonnegative form [35], while the sum of the last 3 terms is a coercive polynomial. The objective is coercive, so  $f_{\min}$  is achieved at a critical point. In computation, we got  $f_{\min} \approx 0.9413$  and a minimizer

$$(0.5632, 0.5632, 0.5632, 0.7510).$$

The computational results for standard Lasserre's relaxations and the new ones (3.8)-(3.9) are in Table 8. It confirms that  $f_k = f_{\min}$  for all  $k \geq 3$ , up to numerical round-off errors.

TABLE 8. Computational results for Example 6.8.

order $k$	Without L.M.E.		With L.M.E.	
	lower bound	time	lower bound	time
2	$-\infty$	0.3984	-0.3360	0.9321
3	$-\infty$	0.7634	0.9413	0.5240
4	$-6.4896 \cdot 10^5$	4.5496	0.9413	1.7192
5	$-3.1645 \cdot 10^3$	24.3665	0.9413	8.1228

**Example 6.9.** Consider the optimization problem

$$\begin{cases} \min_{x \in \mathbb{R}^4} & (x_1 + x_2 + x_3 + x_4 + 1)^2 - 4(x_1x_2 + x_2x_3 + x_3x_4 + x_4 + x_1) \\ \text{s.t.} & 0 \leq x_1, \dots, x_4 \leq 1. \end{cases}$$

The matrix  $L(x)$  is given in Example 4.3. The objective is the dehomogenization of Horn's form [35]. The feasible set is compact, so  $f_{\min}$  is achieved at a critical point. One can show that  $f_{\min} = 0$ . For each  $t \in [0, 1]$ , the point  $(t, 0, 0, 1 - t)$  is a global minimizer. The computational results for standard Lasserre's relaxations and the new ones (3.8)-(3.9) are in Table 9.

TABLE 9. Computational results for Example 6.9.

order	Without L.M.E.		With L.M.E.	
	lower bound	time	lower bound	time
2	-0.0279	0.2262	$-5 \cdot 10^{-6}$	1.1835
3	-0.0005	0.4691	$-6 \cdot 10^{-7}$	1.6566
4	-0.0001	3.1098	$-2 \cdot 10^{-7}$	5.5234
5	$-4 \cdot 10^{-5}$	16.5092	$-6 \cdot 10^{-7}$	19.7320



## 7. DISCUSSIONS

**7.1. Tight relaxations using preorderings.** When the global minimum value  $f_{\min}$  is achieved at a critical point, the problem (1.1) is equivalent to (3.7). We proposed relaxations (3.8)-(3.9) for solving (3.7). Note that

$$\text{IQ}(c_{eq}, c_{in})_{2k} + \text{IQ}(\phi, \psi)_{2k} = \text{Ideal}(c_{eq}, \phi)_{2k} + \text{Qmod}(c_{in}, \psi)_{2k}.$$

If we replace the quadratic module  $\text{Qmod}(c_{in}, \psi)$  by the preordering of  $(c_{in}, \psi)$  [20, 24], we can get further tighter relaxations. For convenience, write  $(c_{in}, \psi)$  as a single tuple  $(g_1, \dots, g_\ell)$ . Its preordering is the set

$$\text{Preord}(c_{in}, \psi) := \sum_{r_1, \dots, r_\ell \in \{0,1\}} g_1^{r_1} \cdots g_\ell^{r_\ell} \Sigma[x].$$

A tighter relaxation than (3.8), of the same order  $k$ , is

$$(7.1) \quad \begin{cases} f_k^{',pre} := \min & \langle f, y \rangle \\ \text{s.t.} & \langle 1, y \rangle = 1, L_{c_{eq}}^{(k)}(y) = 0, L_\phi^{(k)}(y) = 0, \\ & L_{g_1^{r_1} \dots g_\ell^{r_\ell}}^{(k)}(y) \succeq 0 \quad \forall r_1, \dots, r_\ell \in \{0,1\}, \\ & y \in \mathbb{R}^{\mathbb{N}_{2k}^n}. \end{cases}$$

Similar to (3.9), the dual optimization problem of the above is

$$(7.2) \quad \begin{cases} f_k^{pre} := \max & \gamma \\ \text{s.t.} & f - \gamma \in \text{Ideal}(c_{eq}, \phi)_{2k} + \text{Preord}(c_{in}, \psi)_{2k}. \end{cases}$$

An attractive property of the relaxations (7.1)-(7.2) is that: the conclusion of Theorem 3.3 still holds, even if none of the conditions i)-iii) there is satisfied. This gives the following theorem.

**Theorem 7.1.** *Suppose  $\mathcal{K}_c \neq \emptyset$  and Assumption 3.1 holds. Then,*

$$f_k^{pre} = f_k^{',pre} = f_c$$

*for all  $k$  sufficiently large. Therefore, if the minimum value  $f_{\min}$  of (1.1) is achieved at a critical point, then  $f_k^{pre} = f_k^{',pre} = f_{\min}$  for all  $k$  big enough.*

*Proof.* The proof is very similar to the **Case III** of Theorem 3.3. Follow the same argument there. Without Assumption 3.2, we still have  $\hat{f}(x) \equiv 0$  on the set

$$\mathcal{K}_3 := \{x \in \mathbb{R}^n \mid c_{eq}(x) = 0, \phi(x) = 0, c_{in}(x) \geq 0, \psi(x) \geq 0\}.$$

By the Positivstellensatz, there exists an integer  $\ell > 0$  and  $q \in \text{Preord}(c_{in}, \psi)$  such that  $\hat{f}^{2\ell} + q \in \text{Ideal}(c_{eq}, \phi)$ . The resting proof is the same.  $\square$

**7.2. Singular constraining polynomials.** As shown in Proposition 5.1, if the tuple  $c$  of constraining polynomials is nonsingular, then there exists a matrix polynomial  $L(x)$  such that  $L(x)C(x) = I_m$ . (The  $C(x)$  is as in (5.1).) Hence, the Lagrange multiplier  $\lambda$  can be expressed as

$$\lambda = L(x) \begin{bmatrix} \nabla f(x) \\ 0 \end{bmatrix}.$$

However, if  $c$  is not nonsingular, then such  $L(x)$  does not exist. For such cases, how can we express  $\lambda$  in terms of  $x$  for critical pairs  $(x, \lambda)$ ? This question is mostly open, to the best of the author's knowledge.

Moreover, for a nonsingular tuple  $c$ , what is a good degree bound for  $L(x)$ ? When all the constraints are linear, a degree bound is given in Proposition 4.1. However, for nonlinear constraints, a degree bound is not known.

**7.3. Rational representation of Lagrange multipliers.** In (5.1), the Lagrange multiplier vector  $\lambda$  is determined by a linear equation. Naturally, one can get

$$\lambda = \left( C(x)^T C(x) \right)^{-1} C(x)^T \begin{bmatrix} \nabla f(x) \\ 0 \end{bmatrix},$$

when  $C(x)$  has full column rank. This rational representation is expensive for usage, because its denominator is typically a high degree polynomial. However,  $\lambda$  might have rational representations other than the above. Can we find a rational representation whose denominator and numerator have low degrees? If this is possible, the methods for optimizing rational functions [3, 15, 27] can be applied. This is an interesting question for future research.

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